ECON 245 - Problem Set 1

Winter 2024

Problem 1. CES Demand

Consider a consumer whose prefferences are defined over the consumption, c_i, of varieties of manufacturing goods i. Preferences are assumed to take the following 'Dixit-Stiglitz' CES form:

$$U = \left[\sum_{i=1}^{n} a_i^{1/\sigma} c_i^{\frac{\sigma-1}{\sigma}}\right]^{\sigma/(\sigma-1)}, \quad \sigma > 1, a_i > 0 \forall i$$

a. Let the price of a variety i be denoted by p_i. Set up the consumer's expenditure minimization problem and prove that the compensated demand function for a particular variety i is given by:

$$c_i = \frac{a_i p_i^{-\sigma}}{\left[\sum_{i=1}^{n} a_i p_i^{1-\sigma}\right]^{\sigma/(\sigma-1)}} U$$

b. Using your answer to (1), prove that the consumer's expenditure function is as follows:

$$E(P, U) = \sum_{i=1}^{n} p_i c_i = \left[\sum_{i=1}^{n} a_i p_i^{1-\sigma}\right]^{1/(1-\sigma)} U,$$

$$\text{where} \quad P = \left[\sum_{i=1}^{n} a_i p_i^{1-\sigma}\right]^{1/(1-\sigma)} \quad \text{is a price index}$$

c. Use your answer to (2) to obtain the indirect utility function.

d. Use Roy's identity to obtain derive the Marshallian demand function for a particular variety i.

Problem 2. The Armington (1969) model

Consider a world that consists of a discrete set $\mathcal{S} = \{1, 2, ..., N\}$ of locations. Firms in location j produce a distinct variety of a final good under conditions of perfect competition with constant returns to scale, subject to the following labor-only technology: $y_j = A_j \ell_j$, where A_j is a location-specific productivity term and ℓ_j is labor used. There are L_j workers in location j, who each inelastically supply one unit of labor at a competitive wage w_i and have preferences given by:

$$U_j = \left[\sum_{j \in \mathcal{S}} a_{ij}^{1/\sigma} c_{ij}^{(\sigma-1)/\sigma} \right]^{\sigma/(\sigma-1)},$$

where $\sigma > 1$, c_{ij} is the consumption of goods sourced from location i in location j, and $a_{ij} > 0$ is a preference shifter. Finally, we assume that trade between locations is subject to iceberg trade costs $\tau_{ij} \geq 1$; that is, country j must ship $\tau_{ij} \geq 1$ units to country i for one unit to arrive, where $\tau_{ii} = 1 \; \forall i \in \mathcal{S}$.

a. Show that the unit price of location $i's$ variety in location j equals $p_{ij} = \tau_{ij} w_i / A_i$.
b. Use your previous answer to show that total expenditures in location j on goods sourced from country i equal:

$$X_{ij} = p_{ij} c_{ij} = a_{ij} \left(\tau_{ij} w_i / A_i \right)^{1-\sigma} X_j P_j^{\sigma-1},$$

   where $P_j = \left[\sum_i a_{ij} (\tau_{ij} w_i A_i)^{1-\sigma} \right]^{1/(1-\sigma)}$ is the CES price index and X_j denotes total consumption expenditure by consumers in location j.
c. Using your answer to (2), derive the share of country j's expenditures on goods sourced from country i, defined as $\lambda_{ij} = \frac{X_{ij}}{X_j}$.
d. Show that the factor market clearing condition in location i can be expressed in terms of its market shares across export destinations, $\left\{ \lambda_{ij} \right\}_{j \in \mathcal{S}}$, and the distribution of world incomes, $\left\{ w_i L_i \right\}_{i \in \mathcal{S}}$. That is, derive a system of N equilibrium conditions that ensures that factor income in each country is equal to value added.
e. Prove that equilibrium changes in welfare in country i satisfy:

$$d \ln U_i = -\frac{1}{\sigma - 1} d \ln \lambda_{ii} + \frac{1}{\sigma - 1} d \ln \left[A_i^{\sigma-1} a_{ii} \right].$$

f. This part asks you to set up a code to solve the model quantitatively. Assume the following parameters: $N = 10$, $a_{ij} = 1 \forall i, j \in \mathcal{S}$, $\sigma = 4$, $L_i = 1 \forall i \in \mathcal{S}$, and $A_i = i$ (that is

$A_1 = 1, A_2 = 2, \ldots, A_{10} = 10$. Solve for the equilibrium wages of this model when $\tau_{ii} = 1$, $\tau_{ij} = 2 \forall i \neq j$. Then re-compute the equilibrium when $\tau_{ii} = 1, \tau_{ij} = 1 \forall i \neq j$. Verify that the formula for the welfare gains from trade that you derived in the previous question holds. Graph the welfare gains from trade against the productivity in each location and interpret your findings.